# THE INFLUENCE OF TERMS OF HIGHER ORDER OF SMALLNESS ON PERIODIC SOLUTIONS OF QUASI-LINEAR SYSTEMS

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This paper deals with terms of different order of smallness on the right-hand side of the equations and their influence on the existence, the form and the stability of periodic solutions of quasi-linear systems. This problem is analyzed in detail with self-contained and non-self-contained systems with one degree of freedom, using Poincaré's method in its modern form. The analysis of systems with several degrees of freedom does not show any new results and will therefore not be adduced.

**1.** Let us first consider the self-contained systems [1 and 2]

$$x^{\cdot\cdot} + k^2 x = \mu F(x, x^{\cdot}, \mu) \tag{1.1}$$

The function $F(x, x^{\cdot}, \mu)$ is analytic with respect to its arguments, $\mu$ is a small positive parameter, $k$ is a constant. It is assumed that the smallness of the parameter $\mu$ guarantees the existence of periodic solutions each of which is expanded in a series in powers of $\mu$. As is known, the solution of a generating system ($\mu = 0$) depends on one parameter and has the form

$$x_0(t) = A_0 \cos kt \tag{1.2}$$

The periodic solutions can be determined using the initial conditions

$$x(0) = A_0 + \beta, \qquad x^{\cdot}(0) = 0$$

where $\beta$ is a function of $\mu$ with $\beta(0) = 0$.

The solution of Equation (1.1) may be written in the form of Poincaré series in powers of $\beta$ and $\mu$

$$x(t, A_0 + \beta, \mu) = (A_0 + \beta)\cos kt + \sum_{n=1}^{\infty} \left[ C_n(t) + \frac{\partial C_n(t)}{\partial A_0}\beta + \ldots \right] \mu^n \tag{1.3}$$

Let us expand the right-hand side of Equation (1.1) in a series in powers of $\mu$

$$\mu F(x, x^{\cdot}, \mu) = \sum_{n=1}^{\infty} H_n(t)\mu^n, \qquad H_n(t) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}F}{d\mu^{n-1}}\right)_{\mu=\beta=0} \tag{1.4}$$

The values of the first four coefficients of $H_n(t)$ are given in [1]. We note that the explicit dependence of the function $F$ on $\mu$ has an effect on the quantities of $H_s(t)$ starting with $H_2(t)$ .

The functions $C_s(t)$ are defined by Formula

$$C_n(t) = \frac{1}{k}\int_0^t H_n(t')\sin k(t-t')\,dt' \qquad (1.5)$$

The index of the function $C_s(t)$ is equal to the power of $\mu$ , the coefficient of which is $H_s(t)$ in the expansion (1.4). The derivatives of any order of $C_s(t)$ with respect to $t$ and $A_0$ are expressed by $H_s(t)$ and $H_s^{\cdot}(t)$ and their derivatives with respect to $A_0$ .

We will estimate all quantities in the formula for periodic solutions of quasi-linear systems subject to the maximum order of the terms of the expansion of the right-hand sides of the equations, where the terms may still influence the given quantity. Obviously such an estimation may be made with respect to the largest index of the functions $C_s(t)$ and their derivatives with respect to $t$ and $A_0$ on which the given quantity is dependent. The formulation "such a quantity is dependent on the terms of order of smallness s" will mean in the following that the mentioned quantity is generally dependent on the coefficients of the terms of order 1 to sth.

As is known, one of conditions of periodicity defines the period of the unknown solution as an expansion in powers of $\mu$ . The second condition of periodicity leads to the relation

$$\sum_{n=1}^{\infty} M_n(T_0, A_0 + \beta)\mu^n = 0 \qquad \left(T_0 = \frac{2\pi}{k}\right) \qquad (1.6)$$

where $T_0$ is a period of the generating solution.

Let us denote the quantities $M_s(T_0, A_0 + \beta)$ for $\beta = 0$ by $M_s$ and we will obtain

$$M_1 = C_1(T_0) = -\frac{1}{k}\int_0^{T_0} F(x, x^{\cdot}, 0)\sin kt\,dt$$

$$M_2 = C_2(T_0) + \frac{1}{2k^2A_0}{}_1{}^{\cdot 2}(T_0)] \qquad \text{and so on.}$$

The formulas of paper [1] show that the quantity $M_s$ depends on the terms of order $n$ . The basic amplitudes $A_0$ are generally defined by Equation

$$C_1(T_0) = 0 \qquad (1.7)$$

Consequently, in this case the values $A_0$ depend on the terms of first order.

If the multiplicity of the considered root of Equation (1.7) is equal to $l$ , then the expansion of the quantity $\beta$ may assume the form [2]

$$\beta = \sum_{n=1}^{\infty} A_{n/r}\mu^{n/r} \qquad (r = 1, \ldots, l) \qquad (1.8)$$

For $l = 1$ the coefficients of $A_s$ may be found from the linear equations. For instance, for $A_1$ we have

$$A_1\partial C_1 / \partial A_0 + M_2 = 0$$

It is shown from the equations for the remaining coefficients [1] that the coefficient of $A_s$ is dependent on the terms of order $n + 1$ .

For $M_2 \neq 0$ and for the expansion of $\beta$ in fractional powers of $\mu$ , the first coefficient of the expansion is defined by Equation [2]

$$\frac{1}{r!}\frac{\partial^r C_1}{\partial A_0{}^r}A_{1/r}{}^r + M_2 = 0$$

and is consequently also dependent on the terms of second order. An analysis of the remaining equations for the coefficients $A_{s/r}$ shows that these coefficients depend on the terms of order $n+1$. We note, that with the assumed definitions the coefficients with the same index, taken from different types of expansions depend on the terms of different order of smallness. For instance, the coefficient $A_1$ is the first coefficient in the expansion in integer powers of $\mu$ and the $r$th coefficient in the expansion of $\mu^{1/r}$. In the first case $A_1$ depends on the quantities of second order, in the second case $A_1$ depends on the quantities of order $r+1$.

In self-contained systems periodic solutions with constant period will be obtained if the transformation in time is made,

$$t = \tau\left(1 + \sum_{n=r}^{\infty} h_{n/r}\mu^{n/r}\right) \tag{1.9}$$

It may be shown that the quantities of order $(n-r+1)$ may influence the coefficient $h_{n/r}$.

The periodic solution of Equation (1.1) is expanded in the same form of series as the quantity $\beta$

$$x(\tau) = \sum_{n=0}^{\infty} \mu^{n/r} x_{n/r}(\tau) \tag{1.10}$$

The term $A_{n/r}\cos kt$ will be introduced in the formula of the function $x_{n/r}(\tau)$. Therefore the function $x_{n/r}(\tau)$ will generally depend on the terms of order $n+1$.

The stability of periodic solutions of Equations (1.1) for sufficiently small $\mu$ is in the case of simple roots of Equation (1.7) defined by the inequality

$$\partial C_1 / \partial A_0 < 0 \tag{1.11}$$

In the case of two-fold roots of Equation (1.7) and of the expansion of $\beta$ in series in powers of $\mu^{1/2}$ the condition of stability [3] will read

$$A_{n/2}\partial^2 C_1 / \partial A_0^2 < 0 \tag{1.12}$$

Here $A_{n/2}$ is the first coefficient not equal to zero with a fraction as index. In the case of the expansion of $\beta$ in integer powers of $\mu$, the stability condition takes on the form

$$A_n \partial^2 C_1 / \partial A_0^2 + \ldots < 0 \tag{1.13}$$

Here $A_n$ is the first coefficient of the expansion in powers of $\beta$ not equal to zero, while the terms not written here depend on $C_{n+1}(T_0)$.

In the case of three-fold roots in Equation (1.7) for $r=3$, the condition of stability is not dependent on the terms of order larger than 1

$$\partial^3 C_1 / \partial A_0^3 < 0 \tag{1.14}$$

For $r=1$ the terms of order larger than 1 have influence on the stability, while for $r=2$ they may influence in some cases, whereas in other cases the stability condition coincides with (1.14).

It has to be underlined that in those cases where the expressions on the left-hand side of the stability conditions cannot vanish, those conditions are not only sufficient but also necessary for sufficiently small $\mu$.

**2.** Let us now consider non-self-contained systems [4 and 5]

$$x^{\cdot\cdot} + m^2 x = f(t) + \mu F(t, x, x^{\cdot}, \mu) \tag{2.1}$$

The function $F(t,x,x^{\cdot},\mu)$ is analytic with respect to $x, x^{\cdot}, \mu$ and periodic with respect to $t$ with period $2\pi$. The quantity $m$ is an integer, while the $m$th coefficients in the expansion of the function $F(t,x,x^{\cdot},\mu)$ in a Fourier series with respect to $t$ vanish.

The solution of the generating system ($\mu = 0$) depends on two parameters and has the form

$$x_0(t) = A_0 \cos mt + \frac{B_0}{m} \sin mt + \varphi(t) \tag{2.2}$$

The original conditions take on the form

$$x(0) = A_0 + \beta + \varphi(0), \qquad x^{\cdot}(0) = B_0 + \gamma + \varphi^{\cdot}(0)$$

The quantity $\gamma$ has the same properties as $\beta$.

The solution of Equation (2.1) may be represented by a Poincaré series in $\beta$, $\gamma$ and $\mu$ which is analogous to the series (1.3). The functions $C_n(t)$ are defined by Formula (1.5).

The periodicity conditions of the solution lead to two relations [4],

$$\sum_{n=0}^{\infty} L_n \mu^n = 0, \qquad \sum_{n=0}^{\infty} N_n \mu^n = 0 \tag{2.3}$$

where

$$L_n = C_n(2\pi, A_0 + \beta, B_0 + \gamma), \qquad N_n = C_1^{\cdot}(2\pi, A_0 + \beta, B_0 + \gamma)$$

The values $L_n$ and $N_n$ for $\beta = \gamma = 0$ will be denoted in the following by $C_n(2\pi)$ and $C_n^{\cdot}(2\pi)$ or $C_n$ and $C_n^{\cdot}$.

The amplitudes $A_0$ and $B_0$ are generally defined from the systems of equations

$$C_1(2\pi) = 0, \qquad C_1^{\cdot}(2\pi) = 0 \tag{2.4}$$

Thus the quantities $A_0$ and $B_0$ depend on the terms of first order.

In the case of simple roots of Equations (2.4) the determinant

$$\Delta = \frac{\partial C_1}{\partial A_0} \frac{\partial C_1^{\cdot}}{\partial B_0} - \frac{\partial C_1}{\partial B_0} \frac{\partial C_1^{\cdot}}{\partial A_0} \tag{2.5}$$

is not equal to zero. In that case the quantities $\beta$ and $\gamma$ are expanded in series in integer powers of $\mu$. The coefficients $A_n$ and $B_n$ of these series are defined from the systems of linear equations. We obtain, for example, for the definition of $A_1$ and $B_1$

$$\frac{\partial C_1}{\partial A_0} A_1 + \frac{\partial C_1}{\partial B_0} B_1 + C_2 = 0, \qquad \frac{\partial C_1^{\cdot}}{\partial A_0} A_1 + \frac{\partial C_1^{\cdot}}{\partial B_0} B_1 + C_2^{\cdot} = 0$$

An analysis of the equation systems for the following coefficients [4] shows that the coefficients $A_n$ and $B_n$ depend on the quantities of order $n + 1$.

In the case of double roots of Equations (2.4) the determinant $\Delta = 0$ and the quantities $\beta$ and $\gamma$ are expanded in series in powers of $\mu$ or $\mu^{\frac{1}{2}}$. The coefficients $A_{\frac{1}{2}}$ and $B_{\frac{1}{2}}$ are defined by Formulas [5]

$$A_{1/2}{}^2 = \frac{2\Delta_1}{\Delta^*} \frac{\partial C_1}{\partial B_0} \frac{\partial C_1^{\cdot}}{\partial B_0}, \qquad B_{1/2}{}^2 = \frac{2\Delta_1}{\Delta^*} \frac{\partial C_1}{\partial A_0} \frac{\partial C_1^{\cdot}}{\partial A_0}$$

In these formulas the determinant of second order $\Delta^*$ is expressed by the derivatives of first and second order of $C_1(2\pi)$ and $C_1^{\cdot}(2\pi)$ with respect to $A_0$ and $B_0$. The determinant

$$\Delta_1 = \frac{\partial C_1}{\partial B_0} C_2^{\cdot} - \frac{\partial C_1^{\cdot}}{\partial B_0} C_2$$

An analysis of the formulas for other coefficients shows that the coefficients $A_{n/2}$ and $B_{n/2}$ depend on the terms of order $n + 1$.

The solutions $x(t)$ are expanded in series of integer or fractional powers of the parameter $\mu$ subject to the character of the expansion of the quantities $\beta$ and $\gamma$. The estimation of the coefficients of the expansion of $x(t)$ remains the same as that for self-contained systems.

The stability of the non-self-contained systems (2.1) in the case of simple roots of Equations (2.4) is for sufficiently small $\mu$ defined from conditions

$$\Delta > 0, \qquad \frac{\partial C_1}{\partial A_0} + \frac{\partial C_1^{\cdot}}{\partial B_0} < 0 \tag{2.6}$$

Consequently, the stability in that case depends only on the terms of first order.

For double roots of Equations (2.4) the first condition of (2.6) is replaced by the following [6]

$$A_{1/2} \frac{\partial \Delta}{\partial A_0} + B_{1/2} \frac{\partial \Delta}{\partial B_0} > 0 \tag{2.7}$$

In this case the stability depends now on the terms of second order. For $A_{\frac{1}{2}} = B_{\frac{1}{2}} = 0$ the stability depends on the terms of still higher order.

Thus the analysis of self-contained and non-self-contained systems with one degree of freedom shows that the influence of terms of various order on periodic solutions is completely identical for both systems. Proceeding from the modern methods of forming periodic solutions, one may show that for systems with several degrees of freedom the estimations will be the same as for systems with one degree of freedom.

**3.** Let us consider the following problem. There is a self-contained or non-self-contained system with one degree of freedom of the form (1.1) or (2.1). We assume that all periodic solutions of this system are well-known. We add one or more functions with multipliers of second or higher order of $\mu$ to the right-hand side of the equation of the system. Then the right-hand side of the equation will look as follows:

$$\mu F(t, x, x^{\cdot}, \mu) = \mu F_1(t, x, x_1^{\cdot}, \mu) + \mu^2 F_2(\ldots) + \mu^3 F_3(\ldots) + \ldots \tag{3.1}$$

where $F_1$ is the original function and $F_2$, $F_3$ are functions which are independent of $F_1$ and which are generally independent of one another and satisfy the same conditions as the functions $F_1$.

We now consider changes in the periodic solutions (the number , form and stability of the solutions) resulting from the addition of functions $F_2$, $F_3, \ldots$

If we expand the right-hand side of the equation according to Formula (1.4), we obtain

$$H_1(t) = H_{11}(t), \quad H_2(t) = H_{12}(t) + H_{21}(t)$$
$$H_3(t) = H_{13}(t) + H_{22}(t) + H_{31}(t) \quad \text{and so on.} \tag{3.2}$$

The first index of $H_{s,s}(t)$ is an index of the function $F_s$, and the second index is an order index according to the second formula of (1.4).

If we expand the solution of the new system in a Poincaré series, then the functions $C_s(t)$ from this series will satisfy the relations analogous to (3.2)

$$C_1(t) = C_{11}(t), \qquad C_2(t) = C_{12}(t) + C_{21}(t) \quad \text{and so on.} \tag{3.3}$$

Considering the problem mentioned, three fundamental solutions are possible.

1. The basic amplitudes $A_0$ and $B_0$ are completely defined from Equations (1.7) or (2.4), i.e. from the equations which depend only on the quantities of the first order. In that case the values $A_0$ and $B_0$ do not change if terms of higher order are added. There are two possibilities: (a) the roots of the equations of the amplitudes are simple. In that case all periodic solutions are expanded in series in integer powers of $\mu$. The terms of second order cannot change the form of the solutions nor their stability. (b) Among the roots of the equations of the amplitudes there are multiple roots. For these roots the solution may be expanded in integer powers and also in fractional powers of the parameter. By addition of the functions $F_s$ $(s = 2,3,\ldots)$ the form of the solutions may change. The solution in the form of a series in integer powers may turn into a solution which is changed in fractional powers and conversely. The real branches of the solutions may

be transformed to imaginary ones and conversely. The characteristic features of the stability of the solutions may change to the opposite ones.

2. The amplitudes $A_0$ and $B_0$, or some values of these amplituses, are not defined from Equations (1.7) or (2.4). For instance, in case

$$C_1(2\pi) \equiv 0, \quad C_1^{\cdot}(2\pi) \equiv 0$$

is valid for non-self-contained systems or $C_1(2\pi) \equiv 0$ is valid for self-contained systems.

More complicated cases are also possible [7]. For instance, Equations (2.4) have the form

$$C_1(2\pi) = \Phi^{(1)}\Psi = 0, \qquad C_1^{\cdot}(2\pi) = \Phi^{(2)}\Psi = 0 \tag{3.4}$$

A part of the values $A_0$ and $B_0$ is defined by Equations

$$\Phi^{(1)} = 0, \qquad \Phi^{(2)} = 0 \tag{3.5}$$

The values $A_0$ and $B_0$ which correspond to $\Psi = 0$, cannot be defined by system (3.4); in that case the determinant $\Delta$ of the system vanishes. These values $A_0$ and $B_0$ may generally be defined from the following system of equations

$$\Psi = 0, \qquad \Phi^{(1)}C_2^{\cdot}(2\pi) - \Phi^{(2)}C_2(2\pi) = 0 \tag{3.6}$$

In this case the periodic solutions for the values $A_0$ and $B_0$, obtained from Equations (3.6) will depend on the function $F_2$.

In the case when the basic amplitudes $A_0$ and $B_0$ are defined by the equations which result from $C_s(2\pi)$ or $C_s^{\cdot}(2\pi)$, the estimation of all quantities which characterize such periodic solutions will depend only on the functions $F_s$ and the functions with smaller index. If the equations of the amplitudes have multiple roots, the functions $F_{s+1}$ and the functions with higher index may influence the form of the equations and their stability.

3. Some values of the amplitudes $A_0$ and $B_0$ can not be defined by any system of equations. For those values of the amplitudes the periodic solution depends on one or two parameters, and in the case of several degrees of freedom it depends on a certain number of parameters not higher than the number of the degree of freedom. Such systems include, for instance, systems possessing first integrals [8]. Here the periodic solution of these systems has only so many parameters as there exist first integrals.

In the considered case it is possible, by adding the functions $F_s$ $(s=2,3..)$ to keep the family of solutions and of the corresponding isolated solutions and to let periodic solutions completely vanish. If there is, for instance, a conservative self-contained system, then the adding of the function $F_s(x,x^{\cdot},\mu)$ on the right-hand side of the equation destroys the conservatism of the system. Only isolated solutions may be kept from the original family of periodic solutions which depend on one parameter.

Finally we study the following problem as an illustration. Consider a quasi-linear system with one degree of freedom, and let it be a self-contained system (1.1). In this system there will be intoduced a small delay of time $\tau_0$. The system takes on the form

$$x^{\cdot\cdot}(t) + k^2 x(t-\tau_0) = \mu F(x(t-\tau_0), x^{\cdot}(t-\tau_0), \mu) \tag{3.7}$$

This problem was posed by I.A.Riabov in his lecture on May 5th, 1964, in the seminar of the Department Analytic Mechanics of the Institute of Mechanics of the Academy of Sciences of the USSR and it was solved for three special examples (the equation of Van der Pol, the equation of Duffing and the oscillation equation of the vacuum-tube oscillator).

If all functions which depend on $\tau_0$ are expanded in power series in $\tau_0$, then Equation (3.7) may be written in the form

$$x^{\cdot\cdot}(t) + k^2 x(t) = \mu F(x(t), x^{\cdot}(t), \mu) + k^2\tau_0 x^{\cdot}(t) + \ldots \tag{3.8}$$

The subsequent terms are of order $\mu\tau_0$, $\tau_0^2$, and so on. Let the delay of time $\tau_0$ have the order $\mu^s$, i.e.

$$\tau_0 = p\mu^n \tag{3.9}$$

where $p$ is a positive coefficient of order 1. Assuming that the systems (3.7) and (3.8) are equivalent, we consider the influence of the small delay on the periodic solutions of the original system (1.1). Under the assumed conditions the added function of lower order will be

$$F_n = pk^2x^{\cdot}(t)$$

As a result of the calculations we obtain

$$C_{n1}(T_0) = p\pi kA_0 \tag{3.10}$$

For $n = 1$ the delay of the order $\mu$ influences the periodic solutions of the given system equally with the term of first order of the nonlinear function $F$. For $n = 2$ it is, for the sake of the estimation of the influence of the delay, necessary to use the above considerations and estimations. In that case the systems which have the amplitudes $A_0$ as simple roots of Equation (1.7) will keep the number, the form and the stability of the periodic solutions when the delay of time is introduced.

In the case of a conservative original system we have $C_{1n}(T_0) \equiv 0$ for arbitrary $n$. The amplitude $A_0$ for a system with delay will be defined from Equation $C_{n1}(T_0) = 0$, from which it follows $A_0 = 0$. This corresponds to the equilibrium state of the system. Thus the existence of arbitrary small delay in a conservative self-contained system leads to the vanishing of periodic solutions.

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